JOURNAL OF COMPUTATIONAL PHYSICS 8, 45–52 (1971)

Corner Singularities in Elliptic Problems by Finite Element Methods

R. WAIT* AND A. R. MITCHELL

Department of Mathematics, University of Dundee, Scotland

Received July 10, 1970

Finite element methods, using bilinear basis functions supplemented by singular functions, are described for solving elliptic boundary value problems with corner singularities. The procedure of mesh refinement in the finite element method in the neighborhood of a singularity is illustrated with respect to the harmonic mixed boundary value problem of the slit. Numerical results obtained right up to the tip of the slit are compared at selected points in the field with values obtained by dual series and finite difference methods.

1. INTRODUCTION

In the approximate solution of elliptic boundary value problems in two dimensions by piecewise polynomial functions, perhaps the greatest inaccuracies are due to boundary singularities. In such problems, high accuracy cannot be obtained by using approximating subspaces consisting of piecewise polynomial functions only, and these functions must be supplemented with singular functions which correspond to the leading singular terms of the expansions of the exact solution at the particular singular points on the boundary. This approach was first used by Fix [3] for calculating eigenvalues of *L*-shaped membranes where the approximating Hermite subspace spanned by a basis of piecewise bicubic polynomials was supplemented by approximate singular functions. More recently Fix and Wakoff [1, Appendix C] have used this method to calculate the lowest eigenvalues of the following rectangular membranes; a hollow square, a *T*-shaped domain, and an *H*-shaped domain.

In the present paper an extremely simple version of this "method of supplementary singular functions" (viz. piecewise bilinear functions plus appropriate supplementary singular functions) is applied to a singular harmonic mixed boundary-value problem, and the results obtained are shown to be comparable in accuracy with those obtained by other more specialized methods which have been

* Mr. Wait's part of the work was carried out while in receipt of an S.R.C. research grant.

used recently to solve this problem. It is worth pointing out that the method of the present paper is applicable irrespective of the number of singularities of known type in the region. The main object of the present paper is to make a sophisticated new extension of finite element methods more accessible.

2. The Harmonic Mixed Boundary Value Problem

The mixed boundary value problem considered consists of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{2.1}$$

in the square region $-(\pi/2) \leqslant x, y \leqslant \pi/2$ with the slit $y = 0$, $0 \leqslant x \leqslant \pi/2$. The mixed boundary conditions consist of

$$\begin{aligned}
\frac{\partial u}{\partial y}\left(x, \pm\frac{\pi}{2}\right) &= 0, & -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}, \\
u\left(\frac{\pi}{2}, y\right) &= 1000, & 0 < y \leqslant \frac{\pi}{2}, \\
&= 0, & -\frac{\pi}{2} \leqslant y < 0, \\
\frac{\partial u\left(-\frac{\pi}{2}, y\right)}{\partial x} &= 0, & -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}, \\
\frac{\partial u(x, 0)}{\partial y} &= 0, & 0 \leqslant x \leqslant \frac{\pi}{2}.
\end{aligned} \tag{2.2}$$

From the antisymmetry of the problem, it is only necessary to consider half the region, viz. the rectangle $R\,[-(\pi/2) \leqslant x \leqslant \pi/2, 0 \leqslant y \leqslant \pi/2]$, and to add the boundary condition

$$u(x, 0) = 500, \qquad -\frac{\pi}{2} \leqslant x < 0$$

to the original boundary conditions. The modified region and boundary conditions are illustrated in Fig. 1, where we have introduced

$$U = u - 500.$$

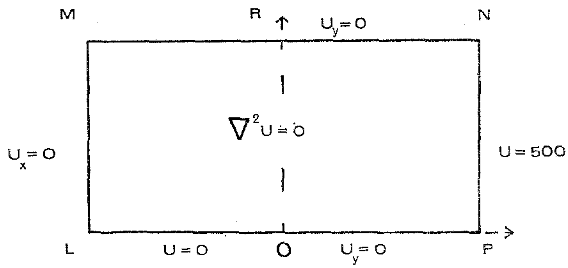

FIG. 1. The modified source problem

3. THE BASIS FUNCTIONS

The region is now divided up into rectangular elements as shown in Fig. 2, with the small region *ABCD*, $\{-(\pi/14) \leqslant x \leqslant \pi/14\} \times \{0 \leqslant y \leqslant \pi/14\}$ containing the singular point 0, further subdivided into thirty two square elements of side $\pi/56$. Two semicricles center 0 and radii $\pi/112$ and $\pi/56$, respectively, are constructed inside *EFGH*, $\{-(\pi/56) \leqslant x \leqslant \pi/56\} \times \{0 \leqslant y \leqslant \pi/56\}$, a small region containing

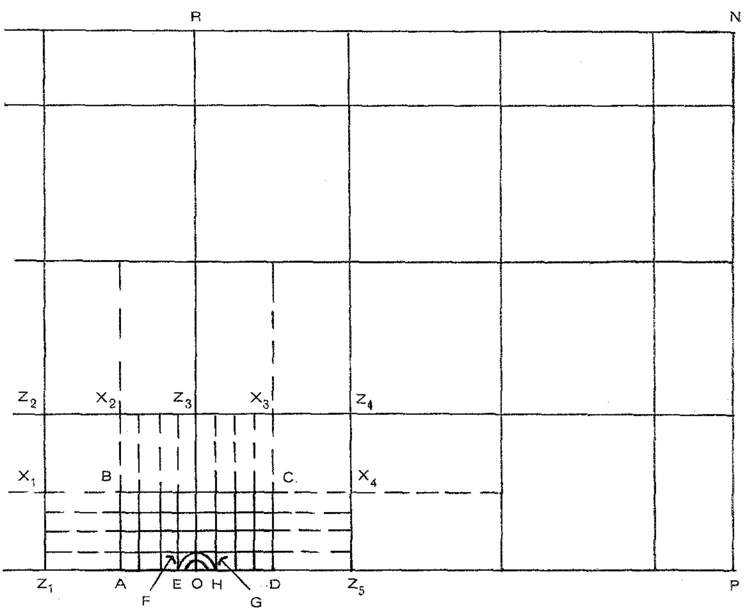

FIG. 2. Part of the grid showing the refinement used near the origin, inside the region *ABCD* $(-\pi/14 \leqslant x \leqslant \pi/14) \times (0 \leqslant y \leqslant \pi/14)$.

0 and lying inside $ABCD$ (see Fig. 2). The procedure for reducing the elements from squares of side $\pi/7$ to squares of side $\pi/56$ inside $ABCD$ is accomplished by drawing in the lines which are signified by dashes in Fig. 2. Since each nodal point on $ABCD$ (except A and D) belongs to four rectangular (or square) elements, the corresponding basis function is composed of four parts and has support only on these four rectangular (or square) elements. The basis functions at the nodes inside $ABCD$ are obtained simply since each node belongs to four equal square elements, whilst those at the nodes outside $ABCD$ remain unchanged except for those at Z_1, Z_2, Z_3, Z_4, and Z_5. The extra grid points introduced due to the refinement are X_1, X_2, X_3, and X_4. The supports of the basis functions at Z_1, Z_2, Z_3, Z_4, Z_5 and X_1, X_2, X_3, X_4 are obvious from Fig. 2.

On the boundary of the original region, there are three distinct types of nodal point:

(i) Nodes where natural boundary conditions are given, and no special basis functions need be constructed.

(ii) Nodes where Dirichlet boundary conditions are given, and at such nodes the coefficients of the respective basis are the given function values.

(iii) The node 0, which is a singular point, and will require singular basis functions which correspond to the leading singular terms of the expansion of the exact solution at the point 0. In this problem, following Lehman [5] and Wigley [9], the solution near the point 0 is of the form

$$U = u - 500 = a_1 r^{1/2} \cos\frac{\theta}{2} + a_2 r \cos\theta + a_3 r^{3/2} \cos\frac{3\theta}{2} + O(r^2) \qquad (3.1)$$

and consequently the first three singular basis functions can be conveniently chosen (Fix [3]), as

$$\begin{aligned} w_1(r, \theta) &= 2^{1/2} R^{1/2} \cos\theta/2, && 0 \leqslant R \leqslant \tfrac{1}{2}, \\ &= 2(R-1)^2 (10R-3) \cos\theta/2, && \tfrac{1}{2} \leqslant R \leqslant 1, \\ &= 0, && R \geqslant 1; \end{aligned} \qquad (3.2)$$

$$\begin{aligned} w_2(r, \theta) &= 2R \cos\theta, && 0 \leqslant R \leqslant \tfrac{1}{2}, \\ &= 2(R-1)^2 (12R-4) \cos\theta, && \tfrac{1}{2} \leqslant R \leqslant 1, \\ &= 0, && R \geqslant 1; \end{aligned} \qquad (3.3)$$

$$\begin{aligned} w_3(r, \theta) &= 2^{3/2} R^{3/2} \cos 3\theta/2, && 0 \leqslant R \leqslant \tfrac{1}{2}, \\ &= 2(R-1)^2 (14R-5) \cos 3\theta/2, && \tfrac{1}{2} \leqslant R \leqslant 1, \\ &= 0, && R \geqslant 1; \end{aligned} \qquad (3.4)$$

where $R = 56\, r/\pi$.

We now consider the expansion

$$V = \sum_{i=1}^{N} a_i w_i , \tag{3.5}$$

where $w_1(r, \theta)$, $w_2(r, \theta)$ and $w_3(r, \theta)$ are given by (3.2), (3.3), and (3.4), respectively, and the basis functions $w_j(x, y)$, $j = 4(1)N$ are chosen to be *bilinear* forms. All the basis functions at grid points inside the complete region shown in Fig. 2 vanish on the boundary of the region, whereas basis functions at nodes on the boundary where Dirichlet conditions are given take the value unity at the node in question, and the value zero at other boundary nodes. No special basis functions are of course required for boundary nodes where Neumann conditions are given.

4. Solution Of The Problem

The construction of the basis functions w_i, $i = 1(1)N$, has been influenced by the shape and number of the elements making up the complete region, the boundary conditions, whether Dirichlet or Neumann, and of course the fact that 0 is a singular point. So far the form of the differential equation (2.1) has not been used. The solution of (2.1) together with natural or Dirichlet boundary conditions is equivalent to minimising the integral

$$I(v) = \iint_R \left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] dR \tag{4.1}$$

over a class of functions $v \in H^1$. We now consider V as given by (3.5) to be an approximation to v, and substitution of (3.5) into (4.1), followed by minimisation with respect to the variable coefficients a_i, leads to an approximate solution of the original problem. It is convenient to keep the integrand in (4.1) in cartesian coordinates for all elements except *EFGH*, where it is simpler to consider it in polar coordinates.

The rate of convergence of the approximate solution V to the exact solution v of this class of problem as $N \to \infty$ has been covered by Fix [4], who shows that if

(1) all basis functions are bilinear functions outside some neighborhood of 0,

(2) coefficients c_i exist such that

$$v - \sum_i c_i w_i = O(r^2), \tag{4.2}$$

throughout some other neighborhood of 0, then

$$\| v - V \|_{W_2^1(R)} = O(h), \tag{4.3}$$

where

$$\| \cdot \|^2_{W_2^1(R)} = \| \cdot \|_{L^2(R)} + I(\cdot),$$

and h is the largest grid spacing. This order of convergence is the same as that obtained by Birkhoff, Schultz, and Varga [2], using only bilinear basis functions in rectangular regions without singularities.

5. Numerical Results

The results obtained at grid points outside the fine mesh region *ABCD* are given in Table I. The results are quoted to three significant figures, and the values obtained using two singular basis functions w_1 and w_2 are identical to four significant figures with those obtained using three singular basis functions w_1, w_2 and w_3. A comparison is made with the values obtained by Whiteman [7] and Motz [6] using dual series and finite difference methods, respectively.

TABLE I

Values calculated at grid points outside the region *ABCD* excluding X_1, X_2, X_3 and X_4. At points at which a comparison was possible the alternative values are (a) Whiteman, (b) Motz.

90	92	110	147	203	276	362	453	500
89(b)								
88	90	108	144	201	275	361	453	
	92(a)	109(a)	145(a)	201(a)	276(a)	362(a)	454(a)	500
	90(b)	107(b)	143(b)	200(b)	276(b)	363(b)	454(b)	
71	72	88	124	183	263	355	451	
	75(a)	90(a)	124(a)	183(a)	265(a)	357(a)	452(a)	500
	73(b)	89(b)	123(b)	182(b)	265(b)	357(b)	452(b)	
39	41	51	78	141	242	347	450	
	43(a)	53(a)	79(a)	141(a)	244(a)	348(a)	450(a)	500
	41(b)	52(b)	78(b)	139(b)	245(b)	349(b)	450(b)	
0	0	0	0	0	227	344	449	
					228(a)	343(a)	448(a)	500
					227(b)	343(b)	449(b)	

The values obtained at grid points inside the fine mesh region *ABCD* are given in Table II. These are quoted to four significant figures, and compared with values obtained by Whiteman [8] using a conformal transformation followed by finite difference methods. The results are quoted for the finite element method using bilinear basis functions supplemented first by two singular basis functions w_1 and and w_2, and then by three singular basis functions w_1, w_2 and w_3. The actual coefficients of the singular basis functions are shown in Table III.

There is no intention in the present paper of obtaining high accuracy results for the particular problem of the slit. These could be obtained by using bicubic instead of bilinear basis functions supplemented by the singular functions $r^{1/2}$,

TABLE II

Values calculated at the grid points inside the region *ABCD*, the lower figure in each pair is that given by Whiteman. The numbers in parentheses indicate the changes in the calculated values when only two singular functions were used.

				(+.1)	(+.1)	(+.1)		
61.4	68.9	78.3	89.7	103.3	118.1	134.3	151.2	168.2
61.9	69.5	78.8	90.2	103.8	119.2	135.7	152.7	169.5
	(+.1)		(+.1)		(+.1)		(+.1)	(+.1)
47.8	54.5	63.4	75.0	89.7	106.9	125.6	144.6	163.1
48.4	55.1	64.0	75.6	90.6	108.3	127.2	146.2	164.6
(+.1)			(+.1)	(+.1)	(+.1)	(+.1)	(+.1)	(+.1)
32.8	38.1	45.6	56.9	73.3	94.1	116.7	138.7	159.0
33.4	38.7	45.6	56.9	74.6	96.2	118.8	140.4	160.4
(+.1)			(+.1)		(+.1)	(+.2)	(+.1)	(+.1)
16.7	19.7	24.2	32.9	51.4	79.6	109.7	134.5	156.3
17.1	20.1	24.8	33.6	53.2	83.7	110.9	136.1	157.5
					(+.4)	(+.1)	(+.1)	(+.1)
0.0	0.0	0.0	0.0	0.0	74.4	106.9	132.9	156.6
					76.4	108.9	134.4	156.5

TABLE III

Coefficients of Singular Basis Functions

No. of singular basis functions	a_1	a_2	a_3
2	11.483	−7.683	—
3	11.324	−7.117	+0.652

$r^{3/2}$ and $r^{5/2}$. This would produce an order of convergence $O(h^3)$ instead of $O(h)$ as given by (4.3) (Fix [4]). The object of the present note is to show how finite element methods supplemented by singular basis functions can cope in a practical manner with elliptic problems with corner singularities. In particular, the refinement of the mesh in the neighborhood of a singularity presents no difficulty from the point of view of construction of basis functions.

In conclusion, it is worth pointing out that the great merit of the finite element method as distinct from other methods such as dual series and finite difference methods for dealing with singularities is that the solution is obtained right up to the singularity. The accuracy of this solution near the singularity depends mainly on the accuracy of the coefficients of the term $r^{1/2}$. In mathematical terms this accuracy is measured by the difference between a_1 in (3.5) and c_1 in (4.2).

REFERENCES

1. G. BIRKHOFF, Piecewise bicubic interpolation and approximation in polygons, *in* "Approximations with Special Emphasis on Spline Functions" (I. J. Schoenberg, Ed.), 185–221, Academic Press, New York, 1969.
2. G. BIRKHOFF, M. H. SCHULTZ, AND R. S. VARGA, Piecewise Hermite interpolation in one and two variables with applications to partial differential equations, *Numer. Mat.* **11** (1968), 232–256.
3. G. FIX, Higher order Rayleigh-Ritz approximations, *J. Math. Mech.* **18** (1969), 645–658.
4. G. FIX, "Bounds and Approximations for Eigenvalues of Self-adjoint Boundary Value Problems," Phd. Thesis, Harvard, 1968.
5. R. S. LEHMAN, Developments at an analytic corner of solutions of elliptic partial differential equations, *J. Maths. Mech.* **8** (1959), 727–760.
6. H. MOTZ, The treatment of singularities in relaxation methods, *Quart. Appl. Math.* **4** (1946), 371–377.
7. J. R. WHITEMAN, Treatment of singularities in a harmonic mixed boundary value problem, *Quart J. Mech. Appl. Math.* **21** (1968), 41–50.
8. J. R. WHITEMAN, Harmonic mixed boundary value problem, U. S. Army MRC Report (1969), 976.
9. N. M. WIGLEY, On a method to subtract off a singularity at a corner for the Dirichlet or Neumann problem, *Math. Comp.* **23** (1969), 395–401.